Journal of Statistical Physics, Vol. 109, Nos. 1/2, October 2002 (© 2002)

Between Equilibrium Fluctuations and Eulerian Scaling: Perturbation of Equilibrium for a Class of Deposition Models

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Received January 7, 2002; accepted April 10, 2002

We investigate propagation of perturbations of equilibrium states for a wide class of 1D interacting particle systems. The class of systems considered incorporates zero range, K-exclusion, misanthropic, "bricklayers" models, and much more. We do not assume attractivity of the interactions. We apply Yau's relative entropy method rather than coupling arguments. The result is *partial extension* of T. Seppäläinen's recent paper. For $0 < \beta < 1/5$ fixed, we prove that, rescaling microscopic space and time by N, respectively $N^{1+\beta}$, the macroscopic evolution of perturbations of microscopic order $N^{-\beta}$ of the equilibrium states is governed by Burgers' equation. The same statement should hold for $0 < \beta < 1/2$ as in Seppäläinen's cited paper, but our method does not seem to work for $\beta \geqslant 1/5$.

KEY WORDS: Hydrodynamic limit; relative entropy; Burgers' equation.

1. INTRODUCTION

In the recent paper,[1] T. Seppäläinen proves that in the so-called totally asymmetric stick process (equivalent to Hammersley's process as seen from a tagged particle), small perturbations of microscopic order $N^{-\beta}$ of equilibrium states, macroscopically propagate according to Burgers' equation, if hydrodynamic limit is taken where space and time are rescaled by N, respectively $N^{1+\beta}$. This result is valid for any $0 < \beta < 1/2$ fixed and goes even beyond the appearence of shocks in the solution of Burgers' equation. Seppäläinen's proof relies on the combinatorial peculiarities of Hammersley's model and on coupling arguments. It is conjectured in ref. 1

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that the result should be valid in much wider context, actually Burgers' equation should govern propagation of disturbances of equilibria (in this scaling regime) for essentially all interacting particle systems with one conserved observable, which under Eulerian scaling lead to a nonlinear 1-conservation law. Seppäläinen's cited result and also our present paper conceptually is closely linked to the work of Esposito *et al.*,[2] where this kind of intermediate scaling was first applied for the simple exclusion model in $d = 3$.

In the present paper we partially extend Seppäläinen's result. We prove a very similar result universally holding for a wide class of interacting particle systems. Our proof is structurally robust, it does not rely on any combinatorial properties of the models considered. We apply Yau's relative entropy method rather than coupling arguments. We pay, of course, a price for this generality: (1) applying the relative entropy method, our results stay valid only up to the emergence of shocks in the Burgers' solution and (2) we can prove our theorem only for $\beta \in (0, 1/5)$ instead of the ideal $\beta \in (0, 1/2)$.

Technically speaking, the proof is a careful application of the relative entropy method. However, we should emphasize that there is some new idea in the "one-block replacement" step, where the standard large deviation argument is replaced by a central limit estimate—and a stronger result is gotten. See Lemma 2 and its proof. Also: since in our scaling regime we have to consider *mesoscopic blocks* of size $N^{2\beta}$ rather than large microscopic blocks, in the one block estimate so-called non-gradient arguments (e.g., spectral gap estimates) are involved.

In our paper we only consider totally asymmetric systems to avoid unnecessary lengthy calculations. All of our results can be extended to partially asymmetric models using the same line of proof.

The paper is organized as follows. In Section 2 we present the models considered and some preliminary computations (infinitesimal generators, equilibria, reversed processes, eulerian hydrodynamic limits, formal perturbations). In Section 3 the main result is precisely formulated in terms of relative entropies. Section 4 contains the proof. This is broken up in several subsections, according to what we consider a logical structure.

2. PRELIMINARIES

2.1. The Models

2.1.1. Notation, State Space

Throughout this paper we denote by $\mathbb{T}^N$ the discrete tori $\mathbb{Z}/N\mathbb{Z}$, $N \in \mathbb{N}$, and by $\mathbb{T}$ the continuous torus $\mathbb{R}/\mathbb{Z}$.

Let $z_{\min}, z_{\max} \in \mathbb{Z} \cup \{-\infty, \infty\}$ with $z_{\min} < z_{\max}$, and $S := [z_{\min}, z_{\max}] \cap \mathbb{Z}$. The state space of the interacting particles system considered is

$$\Omega^N := S^{\mathbb{T}^N}.$$

Configurations will be denoted

$$\underline{z} := (z_j)_{j \in \mathbb{T}^N} \in \Omega^N,$$

2.1.2. Rate Functions, Infinitesimal Generator, and Examples

Following refs. 3–5, we require that the rate function $c: S \times S \to [0, \infty)$ satisfy the following conditions. As we later see, these conditions result that there exists a translation invariant product measure for our Markov process which will play a very important part in our proof.

(A) For any $x, y \in S$

$$c(z_{\min}, y) = 0 = c(x, z_{\max}),$$

Note, that this condition is restrictive only if either $-\infty < z_{\min}$ or $z_{\max} < +\infty$. It guarantees that, with probability 1, the local "spins" z_j stay confined within the bounds $[z_{\min}, z_{\max}]$. In order to avoid degeneracies we also assume that for $x \in (z_{\min}, z_{\max}]$ and $y \in [z_{\min}, z_{\max})$

$$c(x, y) > 0. \tag{1}$$

(B) For any $x, y, z \in S$

$$c(x, y) + c(y, z) + c(z, x) = c(y, x) + c(z, y) + c(x, z).$$

(C) For any $x, y, z \in S \setminus \{z_{\min}\}$

$$c(x, y-1)\, c(y, z-1)\, c(z, x-1) = c(y, x-1)\, c(z, y-1)\, c(x, z-1).$$

This condition is equivalent to requiring that there exist a function $r: S \to [0, \infty)$, with $r(z_{\min}) = 0$, such that for any $x, y \in S \setminus \{z_{\min}\}$

$$\frac{c(x, y-1)}{c(y, x-1)} = \frac{r(x)}{r(y)}.$$

If $-\infty < z_{\min}$ or $z_{\max} < +\infty$, we formally extend r to $\mathbb{Z}$ as $r(x) = 0$ for $x < z_{\min}$, and $r(x) = \infty$ for $x > z_{\max}$.

Remarks. (1) The monotonicity condition $c(x, y+1) \leqslant c(x, y) \leqslant c(x+1, y)$ would imply *attractivity* of the processes defined below. We do not require this property of the rate functions. Our arguments do not rely on coupling ideas.

(2) In the case of unbounded z-variable, $\max\{|z_{\min}|, |z_{\max}|\} = \infty$, we shall also impose some growth condition on the rate function $c(x, y)$. See condition (D) below.

The elementary movements of our Markov process are: $(z_j, z_{j+1}) \to (z_j - 1, z_{j+1} + 1)$ with rate $c(z_j, z_{j+1})$. More formally, we define $\Theta_j \colon \Omega^N \to \Omega^N$:

$$(\Theta_j \underline{z})_i = z_i - \delta_{i,j} + \delta_{i,j+1}.$$

The infinitesimal generator of the process defined on the torus $\mathbb{T}^N$ is

$$L^N f(\underline{z}) = \sum_{j \in \mathbb{T}^N} c(z_j, z_{j+1})(f(\Theta_j \underline{z}) - f(\underline{z})).$$

We shall refer to models defined by this infinitesimal generator as *deposition models*. Our main result is valid for deposition models with

(a) finite state space S with rates $c(x, y)$ satisfying (A)–(C), or

(b) infinite state space S with rates $c(x, y)$ satisfying (A)–(D). (These are either zero range or bricklayers models.)

Clearly, due to the nondegeneracy condition (1), the only conserved quantity of the process is $\sum_j z_j$.

Remark on Notation. Consequently, we shall denote by $\underline{z} = (z_j)_{j \in \mathbb{T}^N}$ an element of the state space Ω^N and by $\underline{\zeta}(s)$ the Markov process on Ω^N with infinitesimal generator L^N.

There are three essentially different classes of examples.

(1) *Bounded Occupation Number.* The only example with $z_{\min} = 0$ and $z_{\max} = 1$ is the *completely asymmetric simple exclusion model*. For any $K > 0$ one can easily check that there exists a finite-parameter family of models with $z_{\min} = 0$ and $z_{\max} = K$ satisfying conditions A to C. These are usually called *generalized K-exclusion models*.

Example. If $z_{\min} = 0$ and $z_{\max} = 2$ with $c(0, y) = 0 = c(x, 2)$ and $c(2, 0) = c(2, 1) + c(1, 0)$, no restriction on $c(1, 1)$ then we have a 3-parameter family of models satisfying conditions A to C. In the case of $z_{\min} = 0$ and $z_{\max} = 3$ one may easily check that we get a 5-parameter family satisfying the conditions.

(2) *Occupation Number Bounded from Below.* There exists an infinite-parameter family of models with $z_{\min} = 0$ and $z_{\max} = +\infty$. In particular, with

$$c(x, y) = r(x) = \mathbb{1}_{\{x>0\}}\, r(x),$$

we get the *zero range models.*

(3) *Unbounded Signed Occupation Number.* From the infinite-parameter family of possible models with $z_{\min} = -\infty$ and $z_{\max} = +\infty$ we point out the following: let $r\colon \mathbb{Z} \to (0, \infty)$ satisfy

$$r(z)\, r(-z+1) = 1.$$

Define

$$c(x, y) = r(x) + r(-y)$$

Following refs. 5–6 we call these models *bricklayers models.*

If the occupation number is not bounded (i.e., the state space is not compact) we need some additional conditions on the growth of the rates. In order to avoid lengthy technical computations we only consider two special cases: the *zero range model* and the *bricklayers* model, defined in examples (2) and (3). For these models we need the following extra conditions which imply that for $x \in \mathbb{N}$ $r(x)$ is essentially linear:

(D) Growth condition for zero range and bricklayers models

(i) $\sup_{x \in \mathbb{N}} |r(x+1) - r(x)| \leqslant a_1 < \infty$.

(ii) There exists $x_0 \in \mathbb{N}$ and $a_2 > 0$ such that $r(x) - r(y) \geqslant a_2$ for all $x \geqslant y + x_0$.

These conditions will guarantee the existence of dynamics and cf. ref. 7 the uniform spectral gap estimate stated in Lemma 5.

2.2. Equilibrium States and Reversed Process

2.2.1. Stationary Measures

From the growth condition D it follows that

$$Z := \sum_{n=1}^{\infty} \prod_{k=1}^{n} r(-k+1) + 1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} r(k)^{-1} < \infty.$$

We define the following probability measure on S

$$\pi(x) := \begin{cases} Z^{-1} \prod_{k=1}^{x} r(k)^{-1} & \text{if} \quad x \geqslant 0, \\ Z^{-1} \prod_{k=1}^{-x} r(-k+1) & \text{if} \quad x \leqslant 0. \end{cases}$$

For $\theta \in \mathbb{R}$ let

$$F(\theta) := \log \sum_{z \in S} e^{\theta z} \pi(z)$$

and

$$\theta_{\min} := \inf\{\theta : F(\theta) < \infty\} \qquad \theta_{\max} := \sup\{\theta : F(\theta) < \infty\}$$

For $\theta \in (\theta_{\min}, \theta_{\max})$ we define the probability measures

$$\pi_\theta(z) := \pi(z) \exp\{\theta z - F(\theta)\}$$

on S. Expectation, variance and covariance with respect to the measure π_θ will be denoted by $\mathbf{E}_\theta(\cdots)$, $\mathbf{Var}_\theta(\cdots)$ and $\mathbf{Cov}_\theta(\cdots)$, respectively.

According to refs. 3–5, conditions A to C guarantee that for any $\theta \in (\theta_{\min}, \theta_{\max})$ the product measure

$$\pi_\theta^N := \prod_{j \in \mathbb{T}^N} \pi_\theta.$$

is stationary for the Markov process. However, due to the conservation of $\sum_j z_j$, on the finite tori T^N these measures are not ergodic. It is a standard matter to check that the measures conditioned on the value of $\sum_j z_j$,

$$\pi_k^N(\underline{z}) := \pi_\theta^N \left(\underline{z} \,\middle|\, \sum_j z_j = k \right), \qquad k \in \mathbb{Z} \cap [N z_{\min}, N z_{\max}],$$

are ergodic. We shall refer to π_θ^N, respectively, π_k^N as *grand canonical*, respectively, *canonical* measures for our model. (The different uses of the subscript should not cause any confusion.)

2.2.2. The Reversed Process

The elementary movements of the reversed process are $(z_{j-1}, z_j) \to (z_{j-1}+1, z_j-1)$ with rate $c(z_j, z_{j-1})$.

Define $\Theta_j^*: \Omega^N \to \Omega^N$,

$$(\Theta_j^* \underline{z})_i = z_i - \delta_{i,j} + \delta_{i,j-1}.$$

The reversed generator on the torus $\mathbb{T}^N$:

$$L^{N*} f(\underline{z}) = \sum_{j \in \mathbb{T}^N} c(z_j, z_{j-1})(f(\Theta_j^* \underline{z}) - f(\underline{z})).$$

Note, that the reveresed process is the same for any π_θ^N, $\theta \in (\theta_{\min}, \theta_{\max})$, or π_k^N, $k \in \mathbb{Z} \cap [Nz_{\min}, Nz_{\max}]$.

2.2.3. Some Expectations

We denote

$$v(\theta) := \mathbf{E}_\theta(z) = \sum_{z \in S} z\pi_\theta(z) = F'(\theta).$$

Elementary computations show

$$v'(\theta) = F''(\theta) = \mathbf{Var}_\theta(z) > 0,$$

thus $(\theta_{\min}, \theta_{\max}) \ni \theta \mapsto v(\theta) \in (z_{\min}, z_{\max})$ is invertible. With some abuse of notation denote the inverse function by $\theta(v)$.

Further notation: we shall denote

$$\Phi_j := c(z_{j+1}, z_j),$$

$$\hat{\Phi}(v) := \mathbf{E}_{\theta(v)}(\Phi_j) = \sum_{x,\, y \in S} \pi_{\theta(v)}(x)\, \pi_{\theta(v)}(y)\, c(x, y).$$

Clearly, if $-\infty < z_{\min} < z_{\max} < \infty$ then $\hat{\Phi}(v)$ is bounded. On the other hand, for the zero range models and bricklayers' models with rate function r satisfying condition (D), straightforward estimates show that

$$\hat{\Phi}(v) \leqslant C|v|$$

and also that Φ_j has finite exponential moment with respect to any grand canonical measure.

Remark on Notation of Finite-Base Cylinder Functions. If Ψ: $S^m \to \mathbb{R}$,then we shall denote $\Psi_j := \Psi(z_j, \ldots, z_{j+m-1})$. The indices $j \in \mathbb{T}^N$ are always meant periodically, mod N. Expectation of Ψ_j with respect to the grand canonical measure $\pi_{\theta(v)}^N$ is denoted

$$\hat{\Psi}(v) := \mathbf{E}_{\theta(v)}(\Psi_j) = \sum_{z_1, \ldots, z_m \in S} \pi_{\theta(v)}(z_1) \cdots \pi_{\theta(v)}(z_m)\, \Psi(z_1, \ldots, z_m).$$

2.3. Hydrodynamic Limits

2.3.1. Eulerian Scaling and Its Formal Perturbation

For the local density $v(t,x)$ of the conserved quantity $\sum_j z_j$, *under Eulerian scaling*, by applying Yau's relative entropy method (see ref. 8, or Chapter 6 of ref. 9, or Section 8 of ref. 10), one gets the pde:

$$\partial_t v + \partial_x \hat{\Phi}(v) = 0. \tag{2}$$

2.3.2. Perturbation of the Euler Equation

Throughout the rest of this paper $v_0 \in (z_{\min}, z_{\max})$ will be fixed and the shorthand notation

$$a_0 := \hat{\Phi}(v_0), \qquad b_0 := \hat{\Phi}'(v_0), \qquad c_0 := \hat{\Phi}''(v_0) \tag{3}$$

will be used. Note that b_0 is the *characteristic speed* for the hyperbolic pde (2), corresponding to v_0. Furthermore, it is assumed that $c_0 \neq 0$.

We now consider a small perturbation of the trivial constant solution $v(t,x) \equiv v_0$ of (2). We fix $\beta > 0$ and insert in (2)

$$v^{(\varepsilon)}(t,x) := v_0 + \varepsilon^\beta u(\varepsilon^{1+\beta} t, \varepsilon(x - b_0 t)).$$

Letting $\varepsilon \to 0$, *formally* the inviscid Burgers' equation is gotten for u:

$$\partial_t u + \frac{c_0}{2}\partial_x(u^2) = 0. \tag{4}$$

3. THE MAIN RESULT

3.1. Further Notation and Terminology

Let $v_0 \in (z_{\min}, z_{\max})$ be fixed and a_0, b_0 and c_0 as defined in (3), $c_0 \neq 0$ is assumed. We also denote $\theta_0 := \theta(v_0)$.

Furthermore, let $u(t,x)$, $t \in [0,T]$, $x \in \mathbb{T}$, be *smooth* solution of Burgers' equation (4) (more precisely: let it be twice continuously differentiable in both variables). We shall use as *absolute reference measure* the stationary measure

$$\pi^N := \prod_{j \in \mathbb{T}^N} \pi_{\theta_0}.$$

We define

$$\theta^N(t,x) := N^\beta(\theta(v_0 + N^{-\beta} u(t, x - N^\beta b_0 t)) - \theta_0)$$

i.e., $\theta(v_0 + N^{-\beta} u(t, x - N^\beta b_0 t)) = \theta_0 + N^{-\beta}\theta^N(t,x)$.

The partial derivatives of $\theta^N(t,x)$ are easily computed:

$$\begin{aligned}\theta_x^N(t,x) &:= \partial_x \theta^N(t,x) = \theta'(v_0 + N^{-\beta} u(t, x - N^\beta b_0 t))\, \partial_x u(t, x - N^\beta b_0 t)\\ \theta_t^N(t,x) &:= \partial_t \theta^N(t,x) = -\theta_x^N(t,x) \times (c_0 u(t, x - N^\beta b_0 t) + N^\beta b_0)\end{aligned} \tag{5}$$

In the computation of $\partial_t \theta^N$ we use the fact that u is smooth solution of (4).

The *time dependent reference measure* (not to be confused with the absolute reference measure!) is

$$\nu_t^N := \prod_{j\in\mathbb{T}^N} \pi_{\theta_0 + N^{-\beta}\theta^N(t,j/N)} = \prod_{j\in\mathbb{T}^N} \pi_{\theta(v_0 + N^{-\beta}u(t, j/N - N^\beta b_0 t))}. \tag{6}$$

The *true distribution* of our process on $\mathbb{T}^N$, at macroscopic time t, i.e., at microscopic time $N^{1+\beta}t$ is

$$\mu_t^N := \mu_0^N \exp\{N^{1+\beta} t L^N\}. \tag{7}$$

The Radon–Nikodym derivatives of these last two probability measures on Ω^N, with respect to the absolute reference measure π^N, are

$$\begin{aligned} f_t^N(\underline{z}) &:= \frac{d\nu_t^N}{d\pi^N}(\underline{z})\\ &= \prod_{j\in\mathbb{T}^N} \exp\{z_j N^{-\beta}\theta^N(t,j/N) - F(\theta_0 + N^{-\beta}\theta^N(t,j/N)) + F(\theta_0)\}\\ h_t^N(\underline{z}) &:= \frac{d\mu_t^N}{d\pi^N}(\underline{z}) = \exp\{N^{1+\beta} t L^{N*}\}\, h_0^N(\underline{z}) \end{aligned} \tag{8}$$

3.2. What Is to Be Proved?

We want to prove that if μ_0^N is close to ν_0^N, in the sense of the relative entropy $H(\mu_0^N \mid \nu_0^N)$ being small, then μ_t^N stays close to ν_t^N in the same sense, uniformly for $t \in [0,T]$.

How close? Given two smooth profiles $u_i\colon \mathbb{T} \to \mathbb{R}$, $i = 1, 2$, let

$$\nu_i^N := \prod_{j\in\mathbb{T}^N} \pi_{\theta(v_0 + N^{-\beta}u_i(j/N))}, \qquad i = 1, 2.$$

Then, an easy computation shows that the relative entropy $H(\nu_2^N \mid \nu_1^N)$ is

$$\begin{aligned} H(\nu_2^N \mid \nu_1^N) &= \sum_{j\in\mathbb{T}^N} H(\pi_{\theta(v_0+N^{-\beta}u_2(j/N))} \mid \pi_{\theta(v_0+N^{-\beta}u_1(j/N))})\\ &= N^{1-2\beta}\theta_0' \int_{\mathbb{T}} (u_2 - u_1)\left(u_2 - \frac{F_0''\theta_0'}{2}(u_2 + u_1)\right) dx + \mathcal{O}(N^{1-3\beta}), \end{aligned}$$

where $\theta_0' := \theta'(v_0)$ and $F_0'' := F''(\theta_0)$. This suggests that one should prove

$$H^N(t) := H(\mu_t^N \mid \nu_t^N) = o(N^{1-2\beta}), \tag{9}$$

uniformly for $t \in [0, T]$.

3.3. Main Result

Consider a deposition model with rate function satisfying conditions A–D (of course, condition D is only needed if S is not finite). Let $v_0 \in (z_{\min}, z_{\max})$ be fixed so that c_0 defined in (3) is nonzero. Let u: $[0, T] \times \mathbb{T} \to \mathbb{R}$ be a *smooth* solution of the inviscid Burgers' equation (4) (more precisely: twice continuously differentiability in both variables suffice). Further on, let ν_t^N, respectively, μ_t^N be the time dependent reference measure, respectively, the true distribution of the mysanthrope process, defined in (6), respectively, (7).

Our main result is the following

Theorem. Let $\beta \in (0, 1/5)$ be fixed. Under the stated conditions, if

$$H(\mu_0^N \mid \pi^N) = \mathcal{O}(N^{1-2\beta})$$

and (9) holds for $t = 0$, than (9) will hold uniformly for $t \in [0, T]$.

Remark. The statement should hold for $\beta < 1/2$, but, with our method, seemingly only $\beta < 1/5$ can be treated.

From this theorem, by applying the entropy inequality the next corollary follows:

Corollary. Under the conditions of the Theorem, for any smooth test function φ: $\mathbb{T} \to \mathbb{R}$

$$N^{-1+\beta} \sum_{j \in \mathbb{T}^N} \varphi((j - N^{1+\beta} b_0 t)/N)(\zeta_j(N^{1+\beta}t) - v_0) \xrightarrow{\mathbf{P}} \int_{\mathbb{T}} \varphi(x)\, u(t, x)\, dx,$$

as $N \to \infty$.

4. PROOF

Our strategy is to get a Grönwall type estimate. We shall prove

$$H^N(t) - H^N(0) \leqslant C \int_0^t H^N(s)\, ds + \mathrm{Err}^N(t). \tag{10}$$

It is assumed that $H^N(0) = o(N^{1-2\beta})$ and the error estimate $\mathrm{Err}^N(t) = o(N^{1-2\beta})$ is the main point.

Important Remark on Further Notation. In the remaining part of the paper, without loss of generality, we assume

$$v_0 = 0, \qquad \theta_0 = 0, \qquad a_0 = 0.$$

This means that from now on z, v, θ, Φ, and $\hat{\Phi}$ stand for $z - v_0$, $v - v_0$, $\theta - \theta_0$, $\Phi - a_0$, and $\hat{\Phi} - a_0$

4.1. Estimating $\partial_t H^N(t)$

In order to prove an inequality like (10) we need to estimate $\partial_t H^N(t)$. Using the well known inequality

$$f L \log f \leqslant L f$$

which holds for every $f \geqslant 0$, straightforward computations lead to

$$\partial_t H^N(t) \leqslant N^{1+\beta} \int_{\Omega^N} \frac{L^{N*} f_t^N}{f_t^N} \, d\mu_t^N - \int_{\Omega^N} \frac{\partial_t f_t^N}{f_t^N} \, d\mu_t^N. \tag{11}$$

(See Chapter 6 of ref. 9 or the paper of ref. 8 for details.)

Further Remarks on Notation. In Sections 4.1 and 4.2 $t \in [0, T]$ will be fixed. In order to avoid heavy notations, in these subsections we do not denote explicitly dependence on t. In particular we shall use the following shorthand notations

$$\theta^N(x) := \theta^N(t, x), \quad \theta_x^N(x) := \theta_x^N(t, x), \quad \theta_t^N(x) := \theta_t^N(t, x),$$

$$u^N(x) := u(t, x - N^\beta b_0 t)$$

Discrete gradient of functions $g \colon \mathbb{T} \to \mathbb{R}$ will be denoted

$$\nabla^N g(x) := N(g(x + 1/N) - g(x)).$$

4.1.1. Computation of $L^{N*} f_t^N / f_t^N$

After straightforward calculations we have

$$\begin{aligned}
\frac{L^{N*}f_t^N}{f_t^N}(\underline{z}) &= \sum_{j\in\mathbb{T}^N}\left(e^{-N^{-1-\beta}(\nabla^N\theta^N)(j/N)}-1\right)\Phi_j \\
&= -N^{-1-\beta}\sum_{j\in\mathbb{T}^N}\theta_x^N(j/N)(\Phi_j-\hat{\Phi}(N^{-\beta}u^N(j/N))) \\
&\quad -N^{-1-\beta}\sum_{j\in\mathbb{T}^N}\theta_x^N(j/N)\,\hat{\Phi}(N^{-\beta}u^N(j/N)) \\
&\quad +\sum_{j\in\mathbb{T}^N}\left(e^{-N^{-1-\beta}(\nabla^N\theta^N)(j/N)}-e^{-N^{-1-\beta}\theta_x^N(j/N)}\right)\Phi_j \\
&\quad +\sum_{j\in\mathbb{T}^N}A(N^{-1-\beta}\theta_x^N(j/N))\,\Phi_j
\end{aligned}$$

where in the last line the shorthand notation $A(x):=e^{-x}-1+x$ is used. The main term is the first sum on the right hand side. We introduce

$$\begin{aligned}
\Psi_j &:= \Phi_j-b_0z_j \\
\hat{\Psi}(v) &:= \mathbf{E}_{\theta(v)}(\Psi_j)=\hat{\Phi}(v)-b_0v
\end{aligned}$$

and write in the main term

$$\Phi_j-\hat{\Phi}(N^{-\beta}u)=(\Psi_j-\hat{\Psi}(N^{-\beta}u))+b_0(z_j-N^{-\beta}u)$$

Thus, eventually we get

$$\begin{aligned}
N^{1+\beta}\int_{\Omega^N}\frac{L^{N*}f_t^N}{f_t^N}\,d\mu_t^N &= -\sum_{j\in\mathbb{T}^N}\theta_x^N(j/N)\int_{\Omega^N}(\Psi_j-\hat{\Psi}(N^{-\beta}u^N(j/N)))\,d\mu_t^N \\
&\quad -b_0\sum_{j\in\mathbb{T}^N}\theta_x^N(j/N)\int_{\Omega^N}(z_j-N^{-\beta}u^N(j/N))\,d\mu_t^N \\
&\quad +\mathrm{Err}_1^N(t)+\mathrm{Err}_2^N(t)+\mathrm{Err}_3^N(t),
\end{aligned} \tag{12}$$

where the error terms are

$$\mathrm{Err}_1^N(t)=-\sum_{j\in\mathbb{T}^N}\theta_x^N(j/N)\,\hat{\Phi}(N^{-\beta}u^N(j/N)), \tag{13}$$

$$\mathrm{Err}_2^N(t)=N^{1+\beta}\sum_{j\in\mathbb{T}^N}\left(e^{-N^{-1-\beta}(\nabla^N\theta^N)(j/N)}-e^{-N^{-1-\beta}\theta_x^N(j/N)}\right)\int_{\Omega^N}\Phi_j\,d\mu_t^N, \tag{14}$$

$$\mathrm{Err}_3^N(t)=N^{1+\beta}\sum_{j\in\mathbb{T}^N}A(N^{-1-\beta}\theta_x^N(j/N))\int_{\Omega^N}\Phi_j\,d\mu_t^N. \tag{15}$$

4.1.2. Computation of $\partial_t f_t^N / f_t^N$

Now we turn our attention to the second term on the right side of (11). From (8) and (5) we get:

$$\frac{\partial_t f_t^N}{f_t^N}(\underline{z}) = \sum_{j\in\mathbb{T}^N} N^{-\beta}\theta_t^N(j/N)(z_j - N^{-\beta}u^N(j/N))$$

$$= -\sum_{j\in\mathbb{T}^N} \theta_x^N(j/N)(c_0 N^{-\beta}u^N(j/N)+b_0)(z_j - N^{-\beta}u^N(j/N))$$

In the last sum we write

$$c_0 N^{-\beta}u = \hat{\Psi}'(N^{-\beta}u) - (\hat{\Psi}'(N^{-\beta}u) - c_0 N^{-\beta}u)$$

and note that the second term is a small error.

Eventually we get:

$$-\int_{\Omega^N} \frac{\partial_t f_t^N}{f_t^N}\, d\mu_t^N(\underline{z})$$

$$= \sum_{j\in\mathbb{T}^N} \theta_x^N(j/N)\,\hat{\Psi}'(N^{-\beta}u^N(j/N)) \int_{\Omega^N} (z_j - N^{-\beta}u^N(j/N))\, d\mu_t^N(\underline{z})$$

$$-b_0 \sum_{j\in\mathbb{T}^N} \theta_x^N(j/N) \int_{\Omega^N} (z_j - N^{-\beta}u^N(j/N))\, d\mu_t^N + \mathrm{Err}_4^N(t) \tag{16}$$

where

$$\mathrm{Err}_4^N(t) = -\sum_{j\in\mathbb{T}^N} \theta_x^N(j/N)(\hat{\Psi}'(N^{-\beta}u^N(j/N))\, c_0 N^{-\beta}u^N(j/N))$$

$$\times \int_{\Omega^N} (z_j - N^{-\beta}u^N(j/N))\, d\mu_t^N(\underline{z}). \tag{17}$$

Note that, when inserting in (11), the second sums on the right hand side of (12) and (16) cancel out.

4.1.3. Blocks

Throughout the paper the one-block size l will be chosen, depending on the system size N, so that asymptotically

$$l \gg N^{2\beta}.$$

We introduce the block averages

$$\Psi_j^l := l^{-1} \sum_{i=0}^{l-1} \Psi_{j+i}, \qquad z_j^l := l^{-1} \sum_{i=0}^{l-1} z_{j+i}.$$

The main terms (i.e., the first sums on the right hand side) in (12), respectively, in (16) become

$$-\sum_{j\in\mathbb{T}^N} \theta_x^N(j/N) \int_{\Omega^N} (\Psi_j^l - \hat{\Psi}(N^{-\beta}u^N(j/N)))\, d\mu_t^N + \mathrm{Err}_5^{N,l}(t), \tag{18}$$

respectively,

$$\sum_{j\in\mathbb{T}^N} \theta_x^N(j/N)\, \hat{\Psi}'(N^{-\beta}u^N(j/N)) \int_{\Omega^N} (z_j^l - N^{-\beta}u^N(j/N))\, d\mu_t^N(\underline{z}) + \mathrm{Err}_6^{N,l}(t). \tag{19}$$

After rearrangement of sums the error terms $\mathrm{Err}_5^{N,l}(t)$, respectively, $\mathrm{Err}_6^{N,l}(t)$ are written as

$$\mathrm{Err}_5^{N,l}(t) = -\sum_{j\in\mathbb{T}^N} \left(l^{-1} \sum_{i=0}^{l-1} \theta_x^N((j-i)/N) - \theta_x^N(j/N) \right) \int_{\Omega^N} \Psi_j\, d\mu_t^N(\underline{z}) \tag{20}$$

$$\begin{aligned} \mathrm{Err}_6^{N,l}(t) = \sum_{j\in\mathbb{T}^N} \Bigg(& l^{-1} \sum_{i=0}^{l-1} \theta_x^N((j-i)/N)\, \hat{\Psi}'(N^{-\beta}u^N((j-i)/N)) \\ & - \theta_x^N(j/N)\, \hat{\Psi}'(N^{-\beta}u^N(j/N)) \Bigg) \int_{\Omega^N} z_j\, d\mu_t^N(\underline{z}). \end{aligned} \tag{21}$$

4.1.4. Sumup and Estimate of the Error Terms (So Far)

Summing up, from (11), (12), (16), (18) and (19), so far we have got:

$$\begin{aligned} \partial_t H^N(t) \leqslant -\sum_{j\in\mathbb{T}^N} \theta_x^N(j/N) \int_{\Omega^N} & (\Psi_j^l - \hat{\Psi}(N^{-\beta}u^N(j/N)) \\ & - \hat{\Psi}'(N^{-\beta}u^N(j/N))(z_j^l - N^{-\beta}u^N(j/N)))\, d\mu_t^N(\underline{z}) \\ & + \mathrm{Err}_1^N(t) + \mathrm{Err}_2^N(t) + \mathrm{Err}_3^N(t) \\ & + \mathrm{Err}_4^N(t) + \mathrm{Err}_5^{N,l}(t) + \mathrm{Err}_6^{N,l}(t) \end{aligned} \tag{22}$$

with the error terms given in (13), (14), (15), (17), (20), (21), respectively.

For the estimate of the these terms we use the following lemma:

Lemma 1. Let $\Psi: \mathbb{Z}^m \to \mathbb{R}$ be a finite cylinder function and denote $\Psi_j := \Psi(z_j, ..., z_{j+m-1})$. Assume that, for $|\gamma| < \gamma_0$, $\mathbf{E}_\pi(\exp\{\gamma\Psi\}) < \infty$. Then there exists a constant $C < \infty$ depending only on m and γ_0 , such that for any $\psi_N: \mathbb{T}^N \to \mathbb{R}$,

$$\sum_{j \in \mathbb{T}^N} \psi_N(j) \int_{\Omega^N} \Psi_j \, d\mu_t^N \leqslant C \max_{j \in \mathbb{T}^N} |\psi_N(j)| \, (N^{1-\beta} + N\mathbf{E}_\pi(\Psi)),$$

uniformly for $t \in [0, T]$.

Proof. We may assume that $\max_{j \in \mathbb{T}^N} |\psi_N(j)| = 1$ and $\mathbf{E}_\pi \Psi(\zeta) = 0$. We set $\gamma_1 := \gamma_0 N^{-\beta} < \gamma_0$ then with the entropy inequality:

$$\left| \sum_{j \in \mathbb{T}^N} \psi_N(j) \int_{\Omega^N} \Psi_j \, d\mu_t^N \right| \leqslant \frac{1}{\gamma_1} H(\mu_t^N \mid \pi^N) + \frac{1}{\gamma_1} \log \mathbf{E}_\pi \exp \left\{ \gamma_1 \sum_{j \in \mathbb{T}^N} \psi_N(j) \, \Psi_j \right\}.$$

Applying the Hölder inequality to the second term, and using that Ψ_j and Ψ_k are independent if $|j-k| > m$ we have

$$\left| \sum_{j \in \mathbb{T}^N} \psi_N(j) \int_{\Omega^N} \Psi_j \, d\mu_t^N \right| \leqslant \frac{1}{\gamma_1} H(\mu_t^N | \pi^N) + \frac{1}{\gamma_1 m} \sum_{j \in \mathbb{T}^N} \Lambda(\gamma_1 m \psi_N(j)),$$

where we use the notation $\Lambda(\gamma) := \log \mathbf{E}_\pi \exp\{\gamma\Psi(\zeta)\}$.

Now, $\Lambda(0) = \Lambda'(0) = 0$, thus we have the asymptotics $\Lambda(\gamma) = \mathcal{O}(\gamma^2)$ for $\|\gamma\| \ll 1$. Since $\max_{j \in \mathbb{T}^N} |\psi_N(j)| = 1$ and $\gamma_1 = \mathcal{O}(N^{-\beta})$ there exists a positive constant C_1 such that $\Lambda(\gamma_1 m \psi_N(j)) \leqslant C_1 \gamma_1^2$ for every $j \in \mathbb{T}^N$. There also exists a constant C_2 with $H(\mu_t^N \mid \pi^N) \leqslant C_2 N^{1-2\beta}$. From these the lemma follows with $C = C_2/\gamma_0 + C_1 \gamma_0 m$. ∎

By Lemma 1 and the smoothness of $u(t, x)$ we readily get:

$$\mathrm{Err}_1^N(t) = \mathcal{O}(N^{1-3\beta}),$$

$$\mathrm{Err}_2^N(t) = \mathcal{O}(N^{-\beta}),$$

$$\mathrm{Err}_3^N(t) = \mathcal{O}(N^{1-4\beta}),$$

$$\mathrm{Err}_4^N(t) = \mathcal{O}(N^{1-4\beta}),$$

$$\mathrm{Err}_5^{N,l}(t) = \mathcal{O}(N^{-\beta} l),$$

$$\mathrm{Err}_6^{N,l}(t) = \mathcal{O}(N^{-2\beta} l).$$

4.2. One Block Replacement

On the right hand side of (22) we replace the block average $\Psi'^{l}_{j}(\underline{z})$ by its "local equilibrium value:" $\hat{\Psi}(z^l_j)$. We denote

$$R(x, y) := \hat{\Psi}(x) - \hat{\Psi}(y) - \hat{\Psi}'(y)(x-y) \tag{23}$$

Then:

$$\begin{aligned} \partial_t H^N(t) &\leqslant - \sum_{j \in \mathbb{T}^N} \theta^N_x(j/N) \int_{\Omega^N} R(z^l_j, N^{-\beta} u^N(j/N))\, d\mu^N_t(\underline{z}) \\ &\quad + M^{N,l}(t) + \mathcal{O}(N^{1-3\beta} \vee N^{-\beta} l), \\ &\leqslant \sup_{\substack{0<t<T \\ j \in \mathbb{T}^N}} |\theta^N_x(j/N)| \sum_{j \in \mathbb{T}^N} \int_{\Omega^N} |R(z^l_j, N^{-\beta} u^N(j/N))|\, d\mu^N_t(\underline{z}) \\ &\quad + M^{N,l}(t) + \mathcal{O}(N^{1-3\beta} \vee N^{-\beta} l), \end{aligned} \tag{24}$$

where

$$M^{N,l}(t) := - \sum_{j \in \mathbb{T}^N} \theta^N_x(j/N) \int_{\Omega^N} (\Psi^l_j - \hat{\Psi}(z^l_j))\, d\mu^N_t(\underline{z}). \tag{25}$$

The estimate of $\int_0^t M^{N,l}(s)\, ds$ is done in the next subsection, by the so-called "one block estimate."

We estimate now the first term on the right hand side of (24). Assume $N = Ml$. By the entropy inequality

$$\begin{aligned} &\sum_{j \in \mathbb{T}^N} \int_{\Omega^N} |R(z^l_j, N^{-\beta} u^N(j/N))|\, d\mu^N_t \\ &\quad \leqslant \frac{1}{\gamma} H(\mu^N_t \mid \nu^N_t) + \frac{1}{\gamma} \log \left(\int_{\Omega^N} \exp\left\{ \gamma \sum_{j \in \mathbb{T}^N} |R(z^l_j, N^{-\beta} u^N(j/N))| \right\} d\nu^N_t(\underline{z}) \right) \end{aligned} \tag{26}$$

We estimate the integral in the second term on the right hand side of (26) using again the Hölder inequality:

$$\begin{aligned} &\int_{\Omega^N} \exp\left\{ \gamma \sum_{j \in \mathbb{T}^N} |R(z^l_j, N^{-\beta} u^N(j/N))| \right\} d\nu^N_t(\underline{z}) \\ &\quad = \int_{\Omega^N} \exp\left\{ \gamma \sum_{i=1}^{l} \sum_{k=0}^{M-1} |R(z^l_{kl+i}, N^{-\beta} u^N((kl+i)/N))| \right\} d\nu^N_t(\underline{z}) \end{aligned}$$

$$\leqslant \left(\prod_{i=1}^{l} \int_{\Omega^N} \exp\left\{ l\gamma \sum_{k=0}^{M-1} |R(z^l_{kl+i}, N^{-\beta} u^N((kl+i)/N))| \right\} d\nu^N_t(\underline{z}) \right)^{1/l}$$

$$= \left(\prod_{j \in \mathbb{T}^N} \int_{\Omega^N} \exp\{ l\gamma \, |R(z^l_j, N^{-\beta} u^N(j/N))| \} \, d\nu^N_t(\underline{z}) \right)^{1/l} \tag{27}$$

In the last setp we use the fact that for any fixed $i \in [1, l]$ the block averages ζ^l_{kl+i}, $k = 0, 1, \ldots, M-1$, are independent under the measure ν^N_t. From (23) it is easy to see that the function

$$x \mapsto R(x + N^{-\beta} u^N(j/N), N^{-\beta} u^N(j/N)) \tag{28}$$

is asymptotically quadratic if $|x| \ll 1$. If the variables $z_i \in S$ are bounded then (28) is automatically bounded. If S is unbounded, but condition D holds, than (28) is asymptotically linearly bounded for $|x| \gg 1$. Thus we may use Lemma 2 stated below, and eventually from (26), (27) we get for γ_0 sufficiently small and $l \geqslant 1/\gamma_0$:

$$\sum_{j \in \mathbb{T}^N} \int_{\Omega^N} |R(z^l_j, N^{-\beta} u^N(j/N))| \, d\mu^N_t(\underline{z}) \leqslant \frac{1}{\gamma_0} H(\mu^N_t \mid \nu^N_t) + CNl^{-1}.$$

Consequently, using this bound in (24) we find

$$\partial_t H^N(t) \leqslant CH^N(t) + M^{N,l}(t) + \mathcal{O}(N^{1-3\beta} \vee N^{-\beta} l \vee N l^{-1}), \tag{29}$$

holding uniformly for $t \in [0, T]$.

Lemma 2. Let $\zeta_1, \zeta_2, \ldots$ be i.i.d. random variables with zero mean. Assume, that for every $\lambda \in \mathbb{R}$

$$\Lambda(\lambda) := \log \mathbf{E}(e^{\lambda \zeta_i}) < \infty. \tag{30}$$

Let the smooth function $G: \mathbb{R} \to \mathbb{R}_+$ be quadratically, respectively, linearly bounded for $|x| \ll 1$, respectively, $|x| \gg 1$, i.e., $G(x) \leqslant C_1(|x| \wedge (x^2/2))$, with some finite constant C_1. Then there exist constants $\gamma_0 > 0$ and $C < \infty$, such that for any $0 < \gamma < \gamma_0$ and $l \geqslant 1/\gamma_0$

$$\mathbf{E} \exp\{\gamma l G((\zeta_1 + \cdots + \zeta_l)/l)\} < C. \tag{31}$$

Remarks. (1) It is worth comparing the statement and proof of Lemma 2 with the corresponding places in previous works applying the one-block replacement, see, e.g., Proposition 1.6. in Part 6. of ref. 9. There usually a weaker statement ($o(l)$ instead of $\mathcal{O}(1)$ on the right hand side of (31)) is gotten by use of more sophisticated tools (large deviation principle instead of central limit estimate). Actually, *we do need* the sharper $\mathcal{O}(1)$ bound.

(2) The statement is easily extended: imposing more restrictive conditions on $\Lambda(\lambda)$, the growth condition on $G(x)$ can be relaxed. E.g., assuming $\Lambda(\lambda) = \mathcal{O}(\lambda^2)$ for $|\lambda| \gg 1$, we may take $G(x)$ quadratically (rather than linearly) bounded at $|x| \gg 1$.

Proof. First we prove the statement with the more restrictive assumption $\Lambda(\lambda) \leqslant C_2\lambda^2/2$. Assume $\gamma < (C_1C_2)^{-1}$ and let ξ be a standard Gaussian random variable, independent of the variables ζ_j. We denote by $\langle \cdots \rangle$ expectation with respect to the variable ξ. Then we have the following chain of (in)equalities:

$$\begin{aligned}
\mathbf{E}\exp\{\gamma l G((\zeta_1 + \cdots + \zeta_l)/l)\} &\leqslant \mathbf{E}\exp\{C_1\gamma\,(\zeta_1 + \cdots + \zeta_l)^2/(2l)\}\\
&= \mathbf{E}\langle \exp(\sqrt{C_1\gamma/l}\,(\zeta_1 + \cdots + \zeta_l)\,\xi\}\rangle\\
&= \langle \mathbf{E}\exp(\sqrt{C_1\gamma/l}\,(\zeta_1 + \cdots + \zeta_l)\,\xi\}\rangle\\
&= \langle \exp\{l\,\Lambda(\sqrt{C_1\gamma/l}\,\xi)\}\rangle\\
&\leqslant \langle \exp\{C_2C_1\gamma\,\xi^2/2)\}\rangle\\
&= (1-\gamma C_1C_2)^{-1/2}.
\end{aligned}$$

Now we consider the general case. Choose α so large, that for any $x \in \mathbb{R}$

$$G(x) < \ln\cosh(\alpha x).$$

One can do this due to the bounds imposed on G. Let $\xi_1, \xi_2, \ldots$ be i.i.d random variables which are also independent of the ζ_j's and have the common distribution $\mathbf{P}(\xi_j = \pm\alpha) = 1/2$. We shall denote by $\langle \cdots \rangle$ expectation with respect to the random variables ξ_j. We choose λ_0, C_3 so that for $|\lambda| < \lambda_0$ the quadratic bound $\Lambda(\lambda) < C_3\lambda^2/2$ holds and fix $\gamma < \lambda_0/\alpha$. Then we have:

$$\begin{aligned}
\mathbf{E}\exp\{\gamma l G((\zeta_1 + \cdots + \zeta_l)/l)\} &\leqslant \mathbf{E}\cosh\,(\alpha\,(\zeta_1 + \cdots + \zeta_l)/l)^{\lceil \gamma l\rceil}\\
&\leqslant \mathbf{E}\langle \exp\{(\xi_1 + \cdots + \xi_{\lceil \gamma l\rceil})(\zeta_1 + \cdots + \zeta_l)/l\}\rangle\\
&= \langle \mathbf{E}\exp\{(\xi_1 + \cdots + \xi_{\lceil \gamma l\rceil})(\zeta_1 + \cdots + \zeta_l)/l\}\rangle\\
&= \langle \exp\{l\,\Lambda((\xi_1 + \cdots + \xi_{\lceil \gamma l\rceil})/l)\}\rangle\\
&\leqslant \langle \exp\{C_3(\xi_1 + \cdots + \xi_{\lceil \gamma l\rceil})^2/(2l)\}\rangle.
\end{aligned}$$

Now, since $\log\cosh(\alpha x) \leqslant \alpha^2 x^2/2$, we can apply to the random variables ξ_j the argument of the first part of this proof, with $C_2 = \alpha^2$ and $C_1 = C_3$, to get

$$\mathbf{E}\exp\{\gamma l G((\zeta_1 + \cdots + \zeta_l)/l)\} \leqslant (1 - \gamma C_3 \alpha^2)^{-1/2}. \qquad \blacksquare$$

4.3. The One Block Estimate

The objective of this section is to provide an estimate for $\int_0^t M^{N,l}(s)\, ds$, where $M^{N,l}(s)$ is given in (25).

4.3.1. Cutoff

We cut off large values of the block averages. In case of compact state space, i.e., $-\infty < z_{\min} < z_{\max} < \infty$ this step is completely omitted. Clearly,

$$M^{N,l}(t) \leqslant A_K^{N,l}(t) + B_K^{N,l}(t), \tag{32}$$

where the terms on the right side are defined as

$$A_K^{N,l}(t) := \sum_{j \in \mathbb{T}^N} \theta_x^N(t, j/N) \int_{\Omega^N} (\Psi_j^l - \hat{\Psi}(z_j^l))\, \mathbb{1}_{\{|z_j^l| \vee \alpha |\Psi_j^l| \leqslant K\}}\, d\mu_t^N(\underline{z}),$$

$$B_K^{N,l}(t) := \sup_{\substack{0 < t < T \\ j \in \mathbb{T}^N}} |\theta_x^N(t, j/N)| \sum_{j \in \mathbb{T}^N} \int_{\Omega^N} |\Psi_j^l - \hat{\Psi}(z_j^l)|\, \mathbb{1}_{\{|z_j^l| \vee \alpha |\Psi_j^l| > K\}}\, d\mu_t^N(\underline{z}),$$

where $\alpha > 0$ is a fixed constant which will only depend on the rate function. For the estimate of $B_K^{N,l}(t)$ we need the following lemma (applied with $m = 1$ or 2 only):

Lemma 3. Let $\Delta: \mathbb{Z}^m \to \mathbb{R}$ be a finite cylinder variable. Then there exists a map $K \mapsto \epsilon(K)$, such that $\lim_{K \to \infty} \epsilon(K) = 0$ and

$$\sum_{j \in \mathbb{T}^N} \int_{\Omega^N} |\Psi_j^l - \hat{\Psi}(z_j^l)|\, \mathbb{1}_{\{|\Delta_j^l| > K\}}\, d\mu_t^N(\underline{z}) \leqslant \epsilon(K)\, N^{1-2\beta}.$$

Proof. The entropy inequality yields:

$$\sum_{j \in \mathbb{T}^N} \int_{\Omega^N} |\Psi_j^l - \hat{\Psi}(z_j^l)|\, \mathbb{1}_{\{|\Delta_j^l| > K\}}\, d\mu_t^N(\underline{z})$$
$$\leqslant \frac{1}{\gamma}\left(H(\mu_t^N \mid \pi^N) + \log \mathbf{E}_{\pi^N} \exp\left\{ \gamma \sum_{j \in \mathbb{T}^N} |\Psi_j^l - \hat{\Psi}(\zeta_j^l)|\, \mathbb{1}_{\{|\Delta_j^l| > K\}} \right\} \right)$$

We note that the jth and kth terms are independent in the last sum if $|j-k| > l+m-1$. By the Hölder inequality, for $l \geqslant m$, we have

$$\log \mathbf{E}_{\pi^N} \exp\left\{\gamma \sum_{j \in \mathbb{T}^N} |\Psi^l_j - \hat{\Psi}(\zeta^l_j)| \, \mathbb{1}_{\{|\Delta^l_j| > K\}}\right\}$$
$$\leqslant N l^{-1} \log \mathbf{E}_{\pi^N} \exp\{2l\gamma \, |\Psi^l - \hat{\Psi}(\zeta^l)| \, \mathbb{1}_{\{|\Delta^l| > K\}}\}.$$

Next we use Cauchy–Schwarz inequality:

$$\mathbf{E}_{\pi^N} \exp\{2l\gamma \, |\Psi^l - \hat{\Psi}(\zeta^l)| \, \mathbb{1}_{\{|\Delta^l| > K\}}\}$$
$$\leqslant 1 + \mathbf{E}_{\pi^N}(\mathbb{1}_{\{|\Delta^l| > K\}} \exp\{2l\gamma \, |\Psi^l - \hat{\Psi}(\zeta^l)|\})$$
$$\leqslant 1 + \{\mathbf{P}_{\pi^N}(|\Delta^l| > K)\}^{1/2} \, \{\mathbf{E}_{\pi^N} \exp\{2l\gamma \, |\Psi^l - \hat{\Psi}(\zeta^l)|\}\}^{1/2}.$$

From standard large deviation arguments it follows that there exists a function $[0, \infty) \ni \gamma \mapsto \Lambda(\gamma) \in [0, \infty)$ (finite for any finite γ!), such that

$$\mathbf{E}_{\pi^N} \exp\{2l\gamma \, |\Psi^l - \hat{\Psi}(\zeta^l)|\} \leqslant \exp\{l \, \Lambda(\gamma)\}.$$

On the other hand, using again a Hölder bound and a standard large deviation estimate, for large l we have

$$\mathbf{P}_{\pi^N}(|\Delta^l| > K) \leqslant m \exp\{-lI(K)/(2m)\},$$

where $x \mapsto I(x)$ is the rate function

$$I(x) := \sup_{\lambda} \, (\lambda x - \log \mathbf{E}_{\pi^N} \exp\{\lambda \Delta\}).$$

We define

$$\gamma(K) := \sup\{\gamma \colon \Lambda(\gamma) < I(K)/(2m)\} \wedge K.$$

Since $\lim_{x \to \infty} I(x) = \infty$, we also have $\lim_{K \to \infty} \gamma(K) = \infty$. Now, putting together all our estimates, we get

$$\sum_{j \in \mathbb{T}^N} \int_{\Omega^N} |\Psi^l_j - \hat{\Psi}(z^l_j)| \, \mathbb{1}_{\{|\Delta^l_j| > K\}} \, d\mu^N_t(\underline{z})$$
$$\leqslant \frac{1}{\gamma(K)} \, (H(\mu^N_t \mid \pi^N) + N l^{-1}(1 + \sqrt{m}\,)).$$

Noting that $H(\mu^N_t \mid \pi^N) = \mathcal{O}(N^{1-2\beta})$ and $l \geqslant C N^{2\beta}$, the lemma follows with $\epsilon(K) = C\gamma(K)^{-1}$. ∎

It is easy to see, that the functions $\Delta_j = z_j$ and $\Delta_j = \Psi_j$ satisfy the conditions of the Lemma 3, thus it follows that there exists a map $K \to \epsilon(K)$ with $\lim_{K\to\infty} \epsilon(K) = 0$ and

$$B_K^{N,l}(t) \leqslant \epsilon(K)\, N^{1-2\beta}. \tag{33}$$

4.3.2. General Tools

We collect in this paragraph the general, *model independent facts* used in the one-block estimate.

Let $\underline{\zeta}(s)$ be a Markov process on the countable state space Ω, with ergodic stationary measure π. Denote by L and L^* the infinitesimal generator and its adjoint, acting on $L^2(\Omega, \pi)$. We denote by $D(f)$ the Dirichlet form associated with the generator L and stationary measure π:

$$D(f) := -\int_\Omega f L f \, d\pi = -\int_\Omega f L^* f \, d\pi$$

The *spectral gap* of the infinitesimal generator L is ρ^{-1} defined by

$$\rho = \rho(L) := \sup_{f \in L^2(\Omega, \pi)} \frac{\mathbf{Var}_\pi(f)}{D(f)} \in (0, \infty].$$

Actually, this means that $(L+L^*)/2$, the symmetric part of L, has a gap of size ρ^{-1} in its spectrum, immediately to the left of the eigenvalue 0.

If $V\colon \Omega \to \mathbb{R}$ is a bounded measurable function we denote

$$\bar{\sigma}(L+V(\cdot)) := \sup\{\text{spectrum of } (L+L^*)/2 + V(\cdot)\}.$$

The following statement is the variational characterization of the "top of the spectrum" of a self-adjoint operator over a Hilbert space. It can be found in any introductory textbook on functional analysis.

Fact 1. For $\bar{\sigma}(L+V(\cdot))$ the following variational formula holds:

$$\bar{\sigma}(L+V(\cdot)) = \sup_h \left(\int_\Omega V(\cdot)\, h \, d\pi - D(\sqrt{h}\,) \right), \tag{34}$$

where the supremum is taken over all probability densities with respect to the stationary measure π.

The second fact is a perturbative estimate of $\bar{\sigma}(L+\varepsilon V(\cdot))$. It can be found, e.g., as Theorem 1.1 in Appendix 3 of ref. 9.

Fact 2. If $V\colon \Omega \to \mathbb{R}$ has zero mean, i.e., $\int_\Omega V\, d\pi = 0$, then, for every $\varepsilon < (2\,\|V\|_\infty\, \rho(L))^{-1}$

$$\bar{\sigma}(L+\varepsilon V(\cdot)) \leqslant \frac{\varepsilon^2 \rho(L)}{1-2\,\|V\|_\infty\, \varepsilon\rho(L)}\, \mathbf{Var}_\pi(V). \tag{35}$$

The third general fact to be used is a direct consequence of the Feynman–Kac formula and straighforward euclidean (inner product) manipulations. Its proof can be found, e.g., in ref. 11 or as Lemma 7.2 in Appendix 1 of ref. 9.

Fact 3. Assume now that $V\colon \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a bounded function. The following bound holds

$$\mathbf{E}_\pi\ \exp\left\{\int_0^t V(s, \underline{\zeta}(s))\, ds\right\} \leqslant \exp\left\{\int_0^t \bar{\sigma}(L+V(s,\cdot))\, ds\right\}, \tag{36}$$

where now $\mathbf{E}_\pi$ denotes expectation over the Markov chain trajectories started from the stationary initial measure π.

4.3.3. Notations

We shall use the notation μ^N, respectively, μ^l for a *generic* probability measure on Ω^N, respectively, Ω^l. We shall denote by $h^N(\underline{z})$, respectively, $h^l(\underline{z})$ their Radon–Nikodym derivatives with respect to the absolute reference measures π^N, respectively, π^l. Further on $\mu^{N,l,j}$ will denote the $[j,\ldots, j+l-1]$ marginal of μ^N and $\mu^{N,l} := N^{-1}\sum_{j\in\mathbb{T}^N}\mu^{N,l,j}$ the average l-dimensional marginal of μ^N. Correspondingly, $h^{N,l,j}(\underline{z})$, respectively, $h^{N,l}(\underline{z})$ will denote the Radon–Nikodym derivatives of $\mu^{N,l,j}$, respectively, $\mu^{N,l}$, with respect to the absolute reference measure π^l

For $k \in \mathbb{Z}$ fixed we denote:

$$\Omega^l_k := \left\{\underline{z} \in \Omega^l\colon \sum_{i=1}^{l} z_i = k\right\},$$

$$m^l_k := \pi^l(\Omega^l_k),$$

$$w^l_k := \mu^l(\Omega^l_k),$$

$$\pi^l_k(\underline{z}) := \pi^l\left(\underline{z}\,\middle|\,\sum_{i=1}^{l} z_i = k\right) = \mathbb{1}_{\{\underline{z}\in\Omega^l_k\}}\,\frac{\pi^l(\underline{z})}{m^l_k},$$

$$\mu^l_k(\underline{z}) := \mu^l\left(\underline{z}\,\middle|\,\sum_{i=1}^{l} z_i = k\right) = \mathbb{1}_{\{\underline{z}\in\Omega^l_k\}}\,\frac{\mu^l(\underline{z})}{w^l_k},$$

$$h^l_k(\underline{z}) := \mathbb{1}_{\{\underline{z}\in\Omega^l_k\}}\,\frac{\mu^l_k(\underline{z})}{\pi^l_k(\underline{z})} = \mathbb{1}_{\{\underline{z}\in\Omega^l_k\}}\,\frac{m^l_k}{w^l_k}\,h^l(\underline{z}).$$

Denote by D^N, D^l respectively D^l_k the following Dirichlet forms

$$D^N(f) := \tfrac{1}{2} \sum_{i=1}^{N} \int_{\Omega^N} c(z_i, z_{i+1})(f(\Theta_i \underline{z}) - f(\underline{z}))^2 \, d\pi^N(\underline{z})$$
$$= \tfrac{1}{2} \sum_{i=1}^{N} \int_{\Omega^N} c(z_i, z_{i-1})(f(\Theta^*_i \underline{z}) - f(\underline{z}))^2 \, d\pi^N(\underline{z})$$
$$D^l(f) := \tfrac{1}{2} \sum_{i=1}^{l-1} \int_{\Omega^l} c(z_i, z_{i+1})(f(\Theta_i \underline{z}) - f(\underline{z}))^2 \, d\pi^l(\underline{z})$$
$$= \tfrac{1}{2} \sum_{i=2}^{l} \int_{\Omega^l} c(z_i, z_{i-1})(f(\Theta^*_i \underline{z}) - f(\underline{z}))^2 \, d\pi^l(\underline{z})$$
$$D^l_k(f) := \tfrac{1}{2} \sum_{i=1}^{l-1} \int_{\Omega^l_k} c(z_i, z_{i+1})(f(\Theta_i \underline{z}) - f(\underline{z}))^2 \, d\pi^l_k(\underline{z})$$
$$= \tfrac{1}{2} \sum_{i=2}^{l} \int_{\Omega^l_k} c(z_i, z_{i-1})(f(\Theta^*_i \underline{z}) - f(\underline{z}))^2 \, d\pi^l_k(\underline{z}).$$

In the definition of D^N *periodic*, in that of D^l and D^l_k *free* boundary conditions are understood.

It is easy to check that for any probability measure μ^l on Ω^l

$$D^l(\sqrt{h^l}\,) = \sum_{k \in \mathbb{Z}} w^l_k D^l_k(\sqrt{h^l_k}\,). \tag{37}$$

Further on, using convexity of the Dirichlet form one can readily prove that

$$D^N(\sqrt{h^N}\,) \geqslant \frac{1}{l} \sum_{j \in \mathbb{T}^N} D^l(\sqrt{h^{N,l,j}}\,). \tag{38}$$

4.3.4. Applying F–K Formula

We return now to the concrete computations. Before the estimate of $\int_0^t A^{N,l}_K(s)\, ds$ we need some more notation (we do not denote explicitly dependence on the cutoff):

$$V^{N,l}_j(\underline{z}) := (\Psi^l_j - \hat{\Psi}(z^l_j))\, \mathbb{1}_{\{|z^l_j| \vee \alpha\, |\Psi^l_j| \leqslant K\}},$$
$$V^l(\underline{z}) := V^{N,l}_1(\underline{z}),$$
$$V^{N,l}_j(t, \underline{z}) := \theta^N_x(N^{-(1+\beta)}t, j/N)\, V^{N,l}_j(\underline{z}),$$
$$V^{N,l}(t, \underline{z}) := \sum_{j \in \mathbb{T}^N} V^{N,l}_j(t, \underline{z}).$$

We denote by $\underline{\zeta}^N(t)$ the Markov process on Ω^N with infinitesimal generator L^N and by $\mathbf{E}_{\mu_0^N}$, respectively, $\mathbf{E}_{\pi^N}$ the *path measure* of this process starting with initial distribution μ_0^N, respectively, π^N.

By the definitions and the entropy inequality we have

$$\int_0^t A_K^{N,l}(s)\,ds$$

$$=\frac{1}{N^{1+\beta}}\mathbf{E}_{\mu_0^N}\left(\int_0^{N^{1+\beta}t} V^{N,l}(s,\underline{\zeta}^N(s))\,ds\right)$$

$$\leqslant\frac{1}{\gamma N^{1+\beta}}\left(H(\mu_0^N\mid\pi^N)+\log\mathbf{E}_{\pi^N}\exp\left\{\int_0^{N^{1+\beta}t}\gamma V^{N,l}(s,\underline{\zeta}^N(s))\,ds\right\}\right).$$

We apply the Feynman–Kac bound (36) and the variational formula (34) to the second term on the right hand side of the last inequality:

$$\log\mathbf{E}_{\pi^N}\exp\left\{\int_0^{N^{1+\beta}t}\gamma V^{N,l}(s,\underline{\zeta}^N(s))\,ds\right\}$$

$$\leqslant\int_0^{N^{1+\beta}t}\bar{\sigma}(L^N+\gamma V^{N,l}(s,\cdot))\,ds$$

$$=\int_0^{N^{1+\beta}t}\sup_{h^N}\left(\int_{\Omega^N}\gamma V^{N,l}(s,\cdot)\,h^N\,d\pi^N-D^N(\sqrt{h^N})\right)ds. \qquad (39)$$

Using (38) we bound the integrand in the last expression

$$\sup_{h^N}\left(\int_{\Omega^N}\gamma V^{N,l}(s,\cdot)\,h^{N,l}\,d\pi^l-D^N(\sqrt{h^N})\right)$$

$$=\sup_{h^N}\left(\sum_{j\in\mathbb{T}^N}\int_{\Omega^l}\gamma V_j^{N,l}(s,\cdot)\,h^{N,l,j}\,d\pi^l-D^N(\sqrt{h^N})\right)$$

$$\leqslant\sup_{h^N}\sum_{j\in\mathbb{T}^N}\left(\int_{\Omega^l}\gamma V_j^{N,l}(s,\cdot)\,h^{N,l,j}\,d\pi^l-\frac{1}{l}D^l(\sqrt{h^{N,l,j}})\right)$$

$$\leqslant\frac{1}{l}\sum_{j\in\mathbb{T}^N}\sup_{h^l}\left(\int_{\Omega^l}l\gamma V_j^{N,l}(s,\cdot)\,h^l\,d\pi^l-D^l(\sqrt{h^l})\right). \qquad (40)$$

Next we use (37) and again the variational formula (34)

$$\begin{aligned}
\sup_{h^l} &\left(\int_{\Omega^l} l\gamma V_j^{N,l}(s,\cdot)\, h^l\, d\pi^l - D^l(\sqrt{h^l}\,)\right) \\
&= \sup_{h^l} \sum_k w_k^l \left(\int_{\Omega_k^l} l\gamma V_j^{N,l}(s,\cdot)\, h_k^l\, d\pi_k^l - D_k^l(\sqrt{h_k^l}\,)\right) \\
&= \sup_{w_\cdot^l} \sum_k w_k^l \sup_{h_k^l} \left(\int_{\Omega_k^l} l\gamma V_j^{N,l}(s,\cdot)\, h_k^l\, d\pi_k^l - D_k^l(\sqrt{h_k^l}\,)\right) \\
&= \sup_{w_\cdot^l} \sum_k w_k^l \bar{\sigma}(L_k^l + l\gamma V_j^{N,l}(s,\cdot)) \\
&= \sup_{w_\cdot^l} \sum_k w_k^l (l\gamma\theta_x^N(s,j/N)\, \mathbf{E}_k^l(V^l) \\
&\qquad + \bar{\sigma}(L_k^l + l\gamma\theta_x^N(s,j/N)(V^l - \mathbf{E}_k^l(V^l)))
\end{aligned} \tag{41}$$

In the first step we used (37). The second step is a straightforward identity. In the third step we have used again (34) and we introduced the notation L_k^l for the infinitesimal generator of the process restricted to Ω_k^l. Finally, in the last step we use the notation introduced at the beginning of the present paragraph.

4.3.5. Spectral Estimates

The rest of the proof of the one block estimate relies on the following three steps: (1) a straightforward estimate of $\mathbf{E}_k^l(V^l)$ and $\mathbf{Var}_k^l(V^l)$; (2) a lower bound of order $\sim l^{-2}$ on the spectral gap of L_k^l, valid uniformly in $k \in \mathbb{Z}$; (3) combining these two and the perturbational bound (35), an upper bound on $\bar{\sigma}(\dots)$ appearing in the last expression.

Lemma 4. There exist constant $C(K) < \infty$ for every $K > K_0$, such that for any l and k the following bounds hold:

$$|\mathbf{E}_k^l(V^l)| \leqslant C(K)\, l^{-1}, \qquad \mathbf{Var}_k^l(V^l) \leqslant C(K)\, l^{-1}. \tag{42}$$

Proof. For $|k| > Kl$ there is nothing to prove, so let $|k| \leqslant Kl$. Restricted on Ω_k^l

$$V^l = \Psi^l - \hat{\Psi}(k/l) - (\Psi^l - \hat{\Psi}(k/l))\, \mathbb{1}_{\{\alpha\, |\Psi^l| > K\}}.$$

Consequently,

$$|\mathbf{E}_k^l(V^l)| \leqslant 2\, |\mathbf{E}_k^l(\Psi^l - \hat{\Psi}(k/l))| + \mathbf{E}_k^l(|\Psi^l - \mathbf{E}_k^l\Psi^l|\, \mathbb{1}_{\{\alpha\, |\Psi^l| > K\}}).$$

By the equivalence of ensembles (see, e.g., Appendix 2. of ref. 9 and also ref. 7)

$$|\mathbf{E}_k^l(\Psi^l - \hat{\Psi}(k/l))| \leqslant C(K)\, l^{-1}.$$

The second term can be estimated with the Cauchy–Schwarz inequality and with large deviation techniques (noting that because of the growth conditions on the rates we can choose such $\alpha > 0$ that $\alpha^{-1}K > |\mathbf{E}_k^l \Psi^l|$ uniformly for $|k| < Kl$) and it can be easily shown to be smaller order then the first one. $\mathbf{Var}_k^l(V^l)$ may be estimated with similar methods. ▮

Lemma 5. There exists a constant $C < \infty$, independent of l and k, such that for any $f \in L^2(\Omega_k^l, \pi_k^l)$

$$\mathbf{Var}_k^l(f) \leqslant C l^2 D_k^l(f). \tag{43}$$

Proof. For the details of the proof of this gap-estimate we refer to refs. 7, 9, and 12. For models with bounded z-variable, $-\infty < z_{\min} < z_{\max} < \infty$, we note that

$$c(x, y) \geqslant \alpha\, r(x)\, \mathbb{1}_{\{x > z_{\min},\, y < z_{\max}\}}.$$

with some positive constant α. Thus, it is sufficient to prove the gap estimate for the *reversible* process with rates $\tilde{c}(x, y) := r(x)\, \mathbb{1}_{\{x > z_{\min},\, y < z_{\max}\}}$, which has the same ergodic stationary measures π_k^l as our original process. For this latter process the induction steps of ref. 7 or Appendix 3 of ref. 9 apply without any essential modification.

In ref. 7 the statement is proved for zero range model with rate function satisfying condition (D). Minor formal (but not essential) modifications of that argument yield the result for the bricklayers' models with rate functions satisfying condition (D). ▮

Remark. Actually we could consider a wider class of models with unbounded spin space, by imposing

$$\inf_y c(x, y) \geqslant \alpha\, r(x)$$

with some positive constant α and $r(x)$ obeying condition (D).

We remark that there exists a constant C depending only on the solution $u(t, x)$ of the Burgers' equation (4), and another constant $C(K)$ which depends also on the cutoff level K, such that

$$\sup_{\substack{0<t<T \\ j\in\mathbb{T}^N}} |\theta^N_x(t, j/N)| \leqslant C, \tag{44}$$

$$\|V^l - \mathbf{E}^l_k V^l\|_\infty \leqslant C(K) \tag{45}$$

Now, combining (35), (42), (43), (44) and (45), we get the following upper bound, which holds for every sufficiently small γ:

$$\bar{\sigma}(L^l_k + l\gamma\theta^N_x(s, j/N)(V^l - \mathbf{E}^l_k(V^l))) \leqslant \frac{C_1(K)\, l^3\gamma^2}{1 - C_2(K)\, \gamma l^3}$$

Setting

$$\gamma := \gamma_0 l^{-3} \qquad \text{with} \quad \gamma_0 < \min\{1, (2C_2(K))^{-1}\}$$

we have

$$\bar{\sigma}(L^l_k + l\gamma\theta^N_x(s, j/N)(V^l - \mathbf{E}^l_k(V^l))) \leqslant C(K)\, \gamma_0^2 l^{-3}.$$

Collecting all the estimates and going backwards through (41), (40), (39), we find eventually

$$\log \mathbf{E}_{\pi^N} \exp\left\{\int_0^{N^{1+\beta}t} \gamma V^{N,l}(s, \underline{\zeta}^N(s))\, ds\right\} \leqslant C(K)\, \gamma_0 N^{2+\beta} l^{-4}$$

and

$$\int_0^t A^{N,l}_K(s)\, ds \leqslant C(K)(N^{-3\beta} l^3 + N l^{-1}) \tag{46}$$

Consequently, from (46), (33) and (32), with any fixed $K < \infty$ we have

$$\int_0^t M^{N,l}(s)\, ds \leqslant \epsilon(K)\, \mathcal{O}(N^{1-2\beta}) + C(K)(N^{-3\beta} l^3 + N l^{-1}) \tag{47}$$

where $C(K)$ is a finite constant which may increase to infinity as $K \to \infty$, and $\epsilon(K) \to 0$ as $K \to \infty$.

4.4. End of Proof

We put together (29) and (47) to get, for any $K < \infty$ fixed (with a C not depending on K)

$$H^N(t) \leqslant H^N(0) + C \int_0^t H^N(s)\, ds + \epsilon(K)\, \mathcal{O}(N^{1-2\beta})$$
$$+ \mathcal{O}(N^{1-3\beta} \vee N^{-\beta} l \vee N l^{-1} \vee N^{-3\beta} l^3).$$

If

$$0 < \beta < \tfrac{1}{5}$$

then we can choose

$$N^{2\beta} \ll l \ll N^{(1+\beta)/3}$$

which ensures

$$\mathcal{O}(N^{1-3\beta} \vee N^{-\beta} l \vee N l^{-1} \vee N^{-3\beta} l^3) = o(N^{1-2\beta}).$$

Thus for every $K < \infty$

$$H^N(t) \leqslant H^N(0) + C \int_0^t H^N(s)\, ds + \epsilon(K)\, N^{1-2\beta} + o(N^{1-2\beta}),$$

where $\lim_{K\to\infty} \epsilon(K) = 0$, and from Grönwall indeed (9) follows, uniformly for $t \in [0, T]$.

Remark. Up to Section 4.3 the calculations for the error terms only imply the restriction $0 < \beta < \frac{1}{3}$, the spectral gap estimate decreases the upper bound to $\frac{1}{5}$. The authors believe that one cannot get better results than $0 < \beta < \frac{1}{3}$ with the methods used.

ACKNOWLEDGMENTS

It is a pleasure of the authors to thank József Fritz many illuminating discussions on the topics of hydrodynamical limits in general and on some particular technical aspects of the present work. B.T. also thanks the kind hospitality of Institut Henri Poincaré, where part of this work was done. This work was partially supported by the Hungarian Scientific Research Fund, Grant T037685.

REFERENCES

1. T. Seppäläinen, Perturbation of the equilibrium for a totally asymmetric stick process in one dimension, *Ann. Probab.* **29**:176–204 (2001).
2. R. Esposito, R. Marra, and H. T. Yau, Diffusive limit of asymmetric simple exclusion, *Rev. Math. Phys.* **6**:1233–1267 (1994).
3. C. Cocozza, Processus des misanthropes, *Z. Wahrscheinlichkeit.* **70**:509–523 (1985).
4. F. Rezakhanlou, Microscopic structure of shocks in one conservation laws, *Ann. I. H. Poincaré—An.* **12**:119–153 (1995).
5. M. Balázs, Growth fluctuations in interface models, *preprint* (2001).

6. M. Balázs, Microscopic structure of the shock in a domain growth model, *J. Stat. Phys.* **105**:511–524 (2001).
7. C. Landim, S. Sethuraman, and S. R. S. Varadhan, Spectral gap for zero range dynamics, *Ann. of Probab.* **24**:1871–1902 (1986).
8. H. T. Yau, Relative entropy and hydrodynamics of Ginzburg–Landau models, *Lett. Math. Phys.* **22**:63–80 (1991).
9. C. Kipnis and C. Landim, *Scaling Limits of Interacting Particle Systems* (Springer, 1999).
10. J. Fritz, *An Introduction to the Theory of Hydrodynamic Limits,* Lectures in Mathematical Sciences (Graduate School of Mathematics, Univ. Tokyo, 2001).
11. K. Komoriya, Hydrodynamic limit for asymmetric mean zero exclusion processes with speed change, *Ann. I. H. Poincaré—Pr.* **34**:767–797 (1998).
12. S. L. Lu and H. T. Yau, Spectral gap and logarothmic Sobolev inequality for Kawasaki and Glauber dynamics, *Comm. Math. Phys.* **156**:399–433 (1993).